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LETTER TO THE EDITOR

Nonlinear resistor fractal networks, topological distances, singly connected bonds and fluctuations

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Abstract. We consider a fractal network of nonlinear resistors, with the voltage V behaving as a power of the current I, $|V| = R|I|^{\alpha}$. The resistance between two points at a distance L is $R(L) \propto L^{\tilde{\zeta}(\alpha)}$. We prove that $\tilde{\zeta}(0)$ describes the scaling of the topological-chemical distance, while $\tilde{\zeta}(\infty)$ describes that of the number of singly connected 'red' bonds. For random resistors, we also consider the width of the resistance distribution, $\Delta R \propto L^{\tilde{\zeta}_2(\alpha)}$. Values for $\tilde{\zeta}$ and $\tilde{\zeta}_2$ are explicitly derived for two model fractals, and $\Delta R/R$ is found to grow with L for the Sierpinski gasket and $\alpha > 1.612$. The relevance of the results to percolation clusters is discussed.

Much of the recent interest in percolation theory has concentrated on identifying geometrical subsets of sites (or bonds) on percolating clusters, which play important roles in determining physical properties. At the percolation threshold, the number of sites (bonds) on each of these subsets scales as a power of the relevant linear scale. It is convenient to consider a finite cluster and to identify two end points (for example, the two points furthest from each other, or the points with largest and smallest z coordinates, etc.). If the Euclidean distance between these end points is L, then for large L the total number of sites on the cluster scales as $M(L) \propto L^D$, where D is the fractal dimensionality (Mandelbrot and Given 1984). Pike and Stanley (1981) also considered the *singly connected* (or 'red' or 'cutting') *bonds*, i.e. bonds whose cutting disconnects the two end sites, and found that their number scales as

$$M_{\text{red}}(L) \propto L^{\tilde{\zeta}_{\text{red}}}. \tag{1}$$

Coniglio (1981, 1982) then proved that

$$\tilde{\zeta}_{\text{red}} = 1/\nu \tag{2}$$

where ν describes the divergence of the correlation length near p_c, $\xi \propto |p - p_c|^{-\nu}$. In two dimensions, $\nu = \frac{4}{3}$ (Nienhuis 1982) and thus $\tilde{\zeta}_{\text{red}} = \frac{3}{4}$.

Coniglio (1981) also showed that $M_{\text{red}}(L)$ determines the low temperature magnetic correlations between two Ising spins at the end points. On the other hand, he related the correlations of Heisenberg spins to the *resistance* between the two end points, which scales as

$$R(L) \propto L^{\tilde{\zeta}_R}. \tag{3}$$

The exponent $\tilde{\zeta}_R$ is directly related to the scaling of the conductivity near p_c. In two dimensions, $\tilde{\zeta}_R \simeq 0.97$ (Zabolitzky 1984, Herrmann *et al* 1984, Hong *et al* 1984, Lobb and Frank 1984).

A third quantity of interest has been the number of bonds on the shortest *topological* (or 'chemical') *path* through the cluster,

$$L_{\text{chem}}(L) \propto L^{\tilde{\zeta}_{\text{chem}}} \tag{4}$$

which relates to the spreading, or growth with time, of the cluster (Alexandrowicz 1980, Grassberger 1985). In two dimensions, $\tilde{\zeta}_{\text{chem}} \simeq 1.15$ (Havlin and Nossal 1984, Hong and Stanley 1983a, b, Vannimenus *et al* 1984).

In an apparently independent context, Kenkel and Straley (1982) introduced the study of *nonlinear resistors*, each with a characteristic voltage-current relation

$$V = r|I|^{\alpha} \operatorname{sgn} I. \tag{5}$$

One may now consider a dilute network of such resistors and describe the generalised nonlinear resistance between the end points by $R(L) \propto L^{\tilde{\zeta}(\alpha)}$. Kenkel and Straley (1982) simulated numerically the average of $R(L)$ for two dilute hierarchical lattices, i.e. the Wheatstone bridge and the 'diamond' (Straley and Kenkel 1984), and estimated the dependence of the exponent $\tilde{\zeta}$ on α†.

This letter has two major objectives. First, we prove that the exponents $\tilde{\zeta}_{\text{red}}$ and $\tilde{\zeta}_{\text{chem}}$ are generally given by the limits $\tilde{\zeta}(\infty)$ and $\tilde{\zeta}(0)$ respectively, of the function $\tilde{\zeta}(\alpha)$. Since $\tilde{\zeta}_R = \tilde{\zeta}(1)$, the function $\tilde{\zeta}(\alpha)$ relates all the three interesting exponents $\tilde{\zeta}_{\text{red}}$, $\tilde{\zeta}_R$ and $\tilde{\zeta}_{\text{chem}}$. Also, since $\alpha = \frac{2}{3}$, the function is relevant for a network of vacuum diodes (Langmuir 1913), and other values of α may represent other useful circuit elements (Kenkel and Straley 1982). Knowledge of the limits $\alpha = 0$ and ∞ is useful in checking further calculations of $\tilde{\zeta}(\alpha)$. It would also be interesting to identify geometrical interpretations for other values of α.

Secondly, we present the first *exact* results for $\tilde{\zeta}(\alpha)$ on two typical fractal structures, i.e. the Mandelbrot-Koch curve (figure 1) and the Sierpinski gasket (figure 2), which have been proposed as models for the infinite incipient cluster (Mandelbrot and Given 1984) or for its backbone (Gefen *et al* 1981) at the percolation threshold. The results are shown in figure 3, and are the same for $\tilde{\zeta}(\alpha)$ and for $\tilde{\zeta}^{\sigma}(\alpha)$. The figure also contains a few values from Kenkel and Straley (1982) and Straley and Kenkel (1984) for comparison.

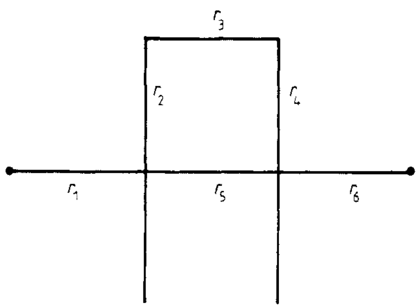

Figure 1. One stage of the Mandelbrot-Koch curve. Each bond is then replaced by a similar structure.

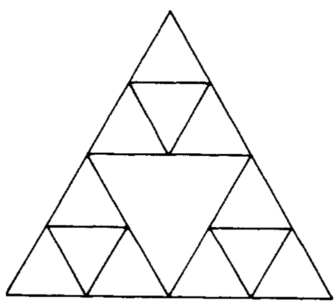

Figure 2. Two stages of iteration of the Sierpinski gasket.

† Actually they considered the average of the conductivity, $\sigma(L) \propto L^{-\tilde{\zeta}^{\sigma}(\alpha)}$. As we discuss elsewhere, $\tilde{\zeta}^{\sigma}(\alpha)$ is not necessarily equal to $\tilde{\zeta}(\alpha)$, and the results may depend on the method of averaging.

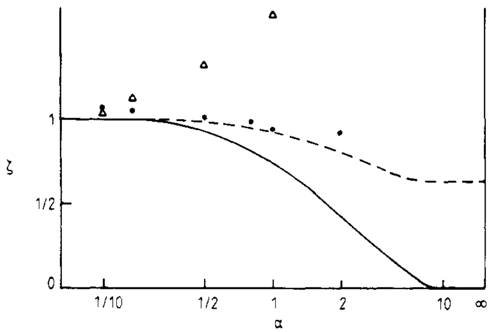

Figure 3. The nonlinear resistivity exponent $\tilde{\zeta}(\alpha)$ for the Sierpinski gasket (full curve) and the Koch curve (broken curve). The dots and the triangles represent data from Kenkel and Straley for the Wheatstone bridge and the diamond hierarchical lattice respectively. The scale of the horizontal axis is $\alpha/(1+\alpha)$.

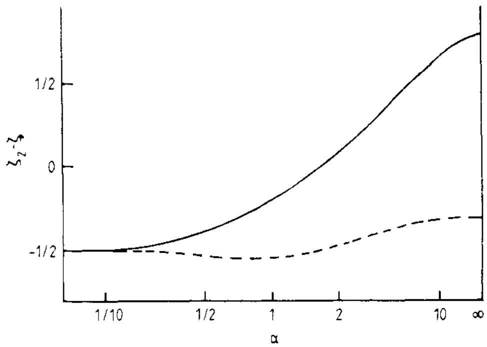

Figure 4. The exponent for the relative width of the resistivity distribution, $(\tilde{\zeta}_2-\tilde{\zeta})$ for the Sierpinski gasket (full curve) and the Koch curve (broken curve). The scale of the horizontal axis is $\alpha/(1+\alpha)$.

In order to check the role played by *randomness*, we allowed a narrow distribution of basic resistivities, with an average $\langle r\rangle$ and a width $\Delta r = (\langle r^2\rangle - \langle r\rangle^2)^{1/2}$. We then studied the scaling of the width of $R(L)$, $\Delta R(L)$, with L, $\Delta R \propto L^{\tilde{\zeta}_2(\alpha)}$. The difference $\tilde{\zeta}_2(\alpha)-\tilde{\zeta}(\alpha)$, which reflects the scaling of the relative width $\Delta R/\langle R\rangle$, is plotted in figure 4. As the figure shows, the Sierpinski gasket has $\tilde{\zeta}_2(\alpha) > \tilde{\zeta}(\alpha)$ for $\alpha > 1.612$, i.e. *the relative fluctuations in R grow faster than the average* $\langle R\rangle$. At least for these cases, this should raise questions on the utility of using only average values in resistivity measurements! As noted recently by Rammal *et al* (1984), ΔR can actually be directly measured via the amplitude of the $1/f$ noise.

We now proceed with a short description of our arguments. It is easy to convince oneself that the resistances defined in (5) add in series as usual, $R_{12}^{\mathrm{ser}} = R_1 + R_2$. On the other hand, the resistance of two resistors in parallel is given by

$$R_{12}^{\mathrm{par}} = (R_1^{-1/\alpha} + R_2^{-1/\alpha})^{-\alpha}. \tag{6}$$

Consider now the example of the Koch curve, figure 1. The total resistance between the end points is given by

$$R = r_1 + r_6 + [r_5^{-1/\alpha} + (r_2 + r_3 + r_4)^{-1/\alpha}]^{-\alpha}. \tag{7}$$

If all the r_i are equal to each other then this reduces to

$$R = [2 + (1+3^{-1/\alpha})^{-\alpha}]r. \tag{8}$$

Since the length scale of R is three times larger than that of r, we may also write $R = 3^{\tilde{\zeta}(\alpha)}r$, i.e.

$$\tilde{\zeta}(\alpha) = \ln[2 + (1+3^{-1/\alpha})^{-\alpha}]/\ln 3. \tag{9}$$

This result is plotted in figure 3.

In the limit $\alpha \to \infty$ we have $(1+3^{-1/\alpha}) \to 2$, and $(1+3^{-1/\alpha})^{-\alpha} \simeq 2^{-\alpha} \to 0$. Thus, $\tilde{\zeta}(\infty) = \ln 2/\ln 3 \simeq 0.6309$. The limiting result $R = 2r$ shows that only the two singly connected 'red' bonds contribute to the resistance, $R = M_{\mathrm{red}}$.

The generalisation of the proof for any 'blob' is straightforward. If the current through the 'blob' is I, and the voltage between its ends is V, then its resistance is given by $R = V/I$. Choosing an arbitrary route from one end of the blob to the other, V can be written as the sum of the voltage drops on each resistor on the route, $V = \Sigma_i V_i = \Sigma_i r_i I_i^\alpha$, so that

$$R = V/I^\alpha = \sum_i r_i (I_i/I)^\alpha. \tag{10}$$

Since the blob is *multi*connected, the current is split into branches, so that $I_i < I$ (there are other resistors, in parallel to i, which carry some of the current). In the limit $\alpha \to \infty$ we thus have $(I_i/I)^\alpha \to 0$, i.e. $R = 0$. *The blobs do not contribute anything to the total resistance* and $R = M_{\text{red}}$.

Consider now the limit $\alpha \to 0$. In this limit the second term in the brackets in (8) becomes $(1+3^{-1/\alpha})^{-\alpha} \to 1$, and the limiting value is equal to the *smaller* of the two resistors which are added in parallel. Thus, $R = 3r$, or $\tilde{\zeta}(0) = 1$. The resistance is equal to the topological-chemical distance, $R = L_{\text{chem}}$.

The proof for several resistors in parallel is straightforward. If the smallest resistor is R_0, then

$$\left(\sum_j R_j^{-1/\alpha}\right)^{-\alpha} = R_0\left[1 + \sum_{j\neq 0}\left(\frac{R_j}{R_0}\right)^{-1/\alpha}\right]^{-\alpha} \xrightarrow[\alpha\to 0]{} R_0. \tag{11}$$

For a more general proof, consider now the first vertex, at one end of the blob. A current I enters into the vertex from the outside, and splits into currents I_i which flow in resistors r_i. We showed that the same result holds for any biterminal graph. If the potential drop on r_i is V_i, then $I_i = (V_i/r_i)^{1/\alpha}$, i.e. $(I_i/I_j) = (V_i r_j / V_j r_i)^{1/\alpha}$. In the limit $\alpha \to 0$ we shall thus have $I_i/I_j \to 0$ whenever $(V_i/r_i) < (V_j/r_j)$. If all the ratios (V_i/r_i) are different from each other then this implies that the whole current flows through the resistor with the largest (V_i/r_i), r_1. The case in which two or more (V_i/r_i) are equal can be treated separately, and yields the same final result. We can now repeat the argument for the vertex at the other end of r_1, and find that the whole current flows through r_2, etc. Finally we identify a single linear chain of resistors, $r_1, r_2, \ldots, r_n$ in the blob, through which the current I flows. The power in the blob is thus $\Sigma_i r_i I^{\alpha+1}$, and this is minimal provided the sum $(\Sigma_i r_i)$ has its smallest possible value. The current thus chooses the topological or chemical shortest route through the blob, and we have $R = L_{\text{chem}}$.

We now return to the Koch curve of figure 1, and consider (7) with a narrow distribution of resistors. Writing $r_i = \langle r\rangle + \delta r_i$, expanding (7) to order δr_i and letting $\langle \delta r_i \, \delta r_j\rangle = (\Delta r)^2 \delta_{ij}$, we find that the average $\langle R(L)\rangle$ relates to $\langle r\rangle$ via (8), while

$$(\Delta R)^2 = \langle R^2\rangle - \langle R\rangle^2 = 3^{2\tilde{\zeta}_2(\alpha)}(\Delta r)^2 \tag{12}$$

with

$$3^{2\tilde{\zeta}_2(\alpha)} = 2 + (1+3^{-1-2/\alpha})/(1+3^{-1/\alpha})^{2(\alpha+1)}. \tag{13}$$

The difference $\tilde{\zeta}_2(\alpha) - \tilde{\zeta}(\alpha)$ is shown in figure 4. Since it is always negative, the distribution of $R(L)$ will come closer and closer to a δ function as the distance L grows.

We now turn to the Sierpinski gasket, figure 2. If all the resistors have the same value, and if a current I enters at one corner and a current I/d exits at each of the other corners, then there is no current through one edge of the central triangle, and we find

$$2^{\tilde{\zeta}(\alpha)} = 1 + [1 + (d-1)2^{-1/\alpha}]^{-\alpha}, \tag{14}$$

with $d = 2$. Equation (14) is easily shown to give the generalisation to the d-dimensional gasket (Gefen *et al* 1981, 1984). The resulting $\tilde{\zeta}(\alpha)$ is also shown in figure 3. In all dimensions we find $\tilde{\zeta}(0) = 1$, indicating the linearity of the chemical route (along an edge), and $\tilde{\zeta}(\infty) = 0$, indicating the absence of any red bonds.

The treatment of the random case is more complicated, and involves repeated application of a generalised triangle-star transformation: the three resistors in figure 5(a) are equivalent to those in figure 5(b) provided

$$r_j = \tfrac{1}{2} \sum_{k=0}^{2} (-1)^k [R_{j+k-1}^{-1/\alpha} + (R_{j+k} + R_{j+k+1})^{-1/\alpha}]^{-\alpha}, \tag{15}$$

with $R_{j+3} = R_j$. This transformation is now repeated several times, as indicated schematically in figure 6. The last step of finding the inverse (star-triangle) transformation, was done to linear order in $\delta r_i = r_i - \langle r \rangle$.

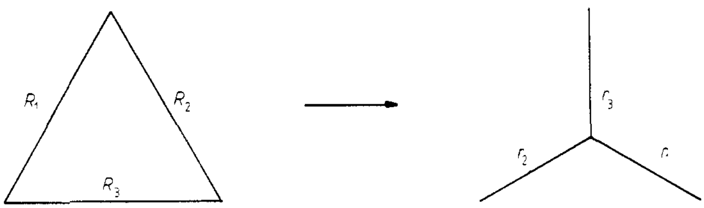

Figure 5. The triangle-star transformation.

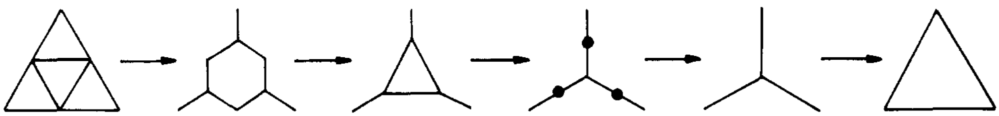

Figure 6. The sequence of star-triangle transformations for deriving the recursion relation for the Sierpinski gasket.

Unlike the case of the Koch curve (or any biterminal renormalised element), the renormalised gasket has *three* terminals (at $d = 2$) and three new resistors. Thus, one generates *nearest-neighbour correlations*, like $\langle \delta r_1 \, \delta r_2 \rangle$, where 1 and 2 are edges of the same triangle. The recursion relations for $\langle \delta r_i^2 \rangle$ and for $\langle \delta r_1 \, \delta r_2 \rangle$ are now coupled. The largest eigenvalue of the appropriate 2×2 matrix is found to be (for $d = 2$)

$$2^{\tilde{\zeta}_2(\alpha)} = \{1 + 2[(2^{1+1/\alpha} - 1)(1 + 2^{1/\alpha})^{-\alpha} + 2^{1/\alpha}]^2\}/(2^{1+1/\alpha} - 1)^2 \tag{16}$$

and the results for $\tilde{\zeta}_2 - \tilde{\zeta}$ are shown in figure 4†. The corresponding eigenvector has equal amplitudes to $\langle \delta r_i^2 \rangle$ and $\langle \delta r_1 \, \delta r_2 \rangle$, indicating that asymptotically one should expect strong correlations.

We note that although $\tilde{\zeta}(\alpha)$ has a similar behaviour for the Koch curve and for the Sierpinski gasket, the details of $(\tilde{\zeta}_2(\alpha) - \tilde{\zeta}(\alpha))$ are quite different. In particular, this difference becomes positive for the gasket at $\alpha > 1.612$. This is probably a direct consequence from the fact that $\tilde{\zeta}(\infty) = 0$, i.e. the gasket contains no red bonds. The smallness of $\tilde{\zeta}(\alpha)$ for large α causes large fluctuations in the resistivity. We note that even in the experimentally accessible case, $\alpha = 1$, $\tilde{\zeta}_2(1) - \tilde{\zeta}(1) \simeq -0.2$ for the gasket, compared to $\tilde{\zeta}_2(1) - \tilde{\zeta}(1) \simeq -0.53$ for the Koch curve.

† Our value of $\tilde{\zeta}_2(1)$ disagrees with that of Rammal *et al* (1984), who ignored the role played by the correlations $\langle \delta r_1 \, \delta r_2 \rangle$.

The Sierpinski gasket was proposed (Gefen *et al* 1981) as the 'loop-within-loop' extreme of the Skal-Shklovskii (1975) model, in which all the links were singly connected. We now identify $(\tilde{\zeta}_2-\tilde{\zeta})$ as an exponent whose magnitude may measure the relative closeness of real systems to this extreme model. Larger values of $(\tilde{\zeta}_2-\tilde{\zeta})$ indicate a larger weight to the loop-within-loop picture. It would be interesting to compare experimental values, either from $1/f$ noise (Rammal *et al* 1984) or from finite size measurements, with our model calculations. Although our results were demonstrated only on model fractals, for which explicit dependences could conveniently be demonstrated, we believe that measurements (or simulations) of $\tilde{\zeta}(\alpha)$ and of $\tilde{\zeta}_2(\alpha)$ on real percolation clusters can yield useful information on their geometrical structure. We are currently also deriving low concentration series for these exponents, and the results confirm our general statements for $\alpha\to 0$ and $\alpha\to\infty$.

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